JOURNAL OF APPROXIMATION THEORY 61, 194–221 (1990)

Unique Continuous Selections for Metric Projections of $C(X)$ onto Finite-Dimensional Vector Subspaces

JÖRG BLATTER

Instituto de Matemática, Universidade Federal do Rio de Janeiro, 21945 Rio de Janeiro, RJ, Brazil

Communicated by Oved Shisha

Received December 22, 1987; revised April 14, 1988

INTRODUCTION

Throughout this paper I deal with best approximation of elements of the space $C(X)$ of all continuous real-valued functions on a compact Hausdorff topological space X in the *uniform norm*

$$\|f\| = \sup\{|f(x)|: x \in X\}, \qquad f \in C(X),$$

by elements of a vector subspace G of $C(X)$ of finite dimension $n \geqslant 1$. For $f \in C(X)$, the *distance of f to G* is the non-negative real number

$$d(f) = \inf\{\|f - g\|: g \in G\},$$

and the *set of best approximations of f in G* is the non-empty compact convex subset

$$P(f) = \{g \in G: \|f - g\| = d(f)\}$$

of G. The (set-valued) *metric projection of* ($C(X)$ onto) G is the mapping P of $C(X)$ into the power set of G which maps $f \in C(X)$ onto $P(f)$, and a *continuous selection for the metric projection P of G* is a continuous mapping S of $C(X)$ into G with the property that $Sf \in P(f)$ for every $f \in C(X)$. G is called a *Chebyshev subspace of* $C(X)$ if every $f \in C(X)$ has a unique best approximation in G, and it is part of the folklore of the subject that in this case the metric projection P of G, considered as a mapping of $C(X)$ into G, is continuous. A. Haar [16] gave the following intrinsic description of Chebyshev subspaces of $C(X)$.

0021-9045/90 $3.00

HAAR'S THEOREM. *G is a Chebyshev subspace of $C(X)$ iff any non-zero function in G has at most $n-1$ distinct zeros.*

Haar already was aware of the fact that the existence of a Chebyshev subspace of $C(X)$ of dimension $n \geqslant 2$ imposes severe restrictions on the underlying space X, and J. C. Mairhuber [19], K. Sieckłucki [23], P. C. Curtis [11], and I. J. Schoenberg and C. T. Yang [22] proved

MAIRHUBER'S THEOREM. *The set of integers $n \geqslant 1$ with the property that $C(X)$ contains an n-dimensional Chebyshev subspace is*

(i) $\{1, ..., k\}$ *if X has only a finite number k of points;*

(ii) $\{1, 2, 3, ...\}$ *if X is homeomorphic to an infinite closed subspace of the unit interval* $[0, 1] = \{t \in \mathbb{R}: 0 \leqslant t \leqslant 1\}$;

(iii) $\{1, 3, 5, ...\}$ *if X is homeomorphic to the unit sphere* $S^1 = \{(r, s) \in \mathbb{R}^2: (r^2 + s^2)^{1/2} = 1\}$; *and*

(iv) $\{1\}$ *in all other cases.*

The purpose of this paper is to extend both Haar's theorem and Mairhuber's theorem from the class of Chebyshev subspaces of $C(X)$ to the larger class of subspaces of $C(X)$ whose metric projection has a unique continuous selection. This is done in Section 1 and Section 2. In the extension of Mairhuber's theorem, universal spaces other than $[0, 1]$ and S^1 appear. These spaces—intervals with split points—are defined in the Appendix; it is also shown in the Appendix that intervals with split points play an important role in areas other than Approximation Theory.

Now a few words on the origins of these results are in order. A. J. Lazar, D. E. Wulbert, and P. D. Morris [17], when dealing with continuous selections for the metric projection P of $C(X)$ onto G, found it interesting to consider the case that G is a *Z-subspace of* $C(X)$, i.e., the interior $\operatorname{int} Z(g)$ of the *zero-set* $Z(g)$ of any non-zero function g in G is empty. They were unaware of their misfortune to have mixed up a relevant approximation-theoretic property of G with an entirely unrelated topological property of X: A. L. Garkavi [15], in a Russian paper which had appeared four years earlier but whose English translation was not to appear for another year (see, however, the announcement of the main results of [15] in [13]), had already studied the case that G is an *almost Chebyshev subspace of* $C(X)$, i.e., the set of functions in $C(X)$ which do not have a unique best approximation in G is of the first category in $C(X)$, and his description of such subspaces of $C(X)$ (see Section 1) implies immediately that

if $\dim G = 1$ (dim = dimension of), then G is a Z-subspace of $C(X)$ iff G is almost Chebyshev; and if $\dim G \geqslant 2$, then G is a Z-subspace of $C(X)$ iff G is almost Chebyshev and X has no isolated points.

The first "victim" of this mix-up was A. L. Brown when he proved in [9] that if G is a Z-subspace of $C(X)$, then if P has a continuous selection it is unique: This result is trivial once one knows that Z-subspaces of $C(X)$ are always almost Chebyshev (see the proposition in Section 1). And Brown insisted when he proved in [10] that if G is a Z-subspace of $C(X)$ of dimension $n \geqslant 2$ whose metric projection has a continuous selection, then if X is metrizable, X is homeomorphic to a subspace of the unit sphere S^1, and if X is not metrizable, X is homeomorphic to a subspace of an interval with split points and G is 1-Chebyshev (see Section 2) but not Chebyshev: This result is not "an extension to Mairhuber's theorem," as Brown claims in the title of his paper, because not every Chebyshev subspace of $C(X)$ is a Z-subspace, and to call it "an extension of Mairhuber's theorem for Z-subspaces (whose metric projection has a continuous selection)," as Brown disclaims the title of his paper in the introduction, seems a bit awkward; I think that, if anything, it should be called an extension of Mairhuber's theorem for spaces without isolated points. In any event, I owe to this very paper of Brown's the idea that an extension of Mairhuber's theorem should indeed exist, and in particular the idea that intervals with split points should appear as new universal spaces in this extension. The latter idea had a special appeal to me because of one interval with split points I had encountered previously as the Gelfand space of the Banach algebra of all regulated real-valued functions on the unit interval [0, 1] (see, e.g., [3]), and the first thing I did upon reading Brown's paper was to prove the results contained in the Appendix of the present paper, namely that intervals with split points can be interpreted as precisely the Gelfand spaces of certain algebras of regulated real-valued functions on the unit interval or, alternatively, as all the order compactifications of the unit interval itself. These results along with the conjecture that Mairhuber's theorem had an extension to subspaces of $C(X)$ whose metric projection has a unique continuous selection I announced in [4] at the 22nd meeting of the Brazilian Analysis Seminar in November, 1985. I did not take up work on the proof of this conjecture, however, until about a year later when, during a course on Approximation Theory I gave at the Federal University of Rio de Janeiro, I discovered a missing link: The metric projection P of $C(X)$ onto G has a unique continuous selection (if and) only if it has a continuous selection and G is almost Chebyshev (see the proposition in Section 1). With this information at hand, I proved that the conditions (1)–(3) in the theorem in Section 1 are necessary for the metric projection P of G to have a unique continuous selection. I then proved the theorem in Section 2 using in the embedding part of the proof only the conditions (1)–(3) on G. And I intended to prove that these conditions are also sufficient for the metric projection P of G to have a unique continuous selection exploiting precisely the extra information on the underlying space X thus obtained from (the proof of) the theorem in Section 2, when I was surprised by a preprint of W. Li's paper [18]. I saw immediately that using Li's main result (see Section 1) I could complete the proof of the theorem in Section 1 without recourse to the theorem in Section 2, and therefore announced the two theorems in [5]. When writing up the present paper, I decided to make the proof of the theorem in Section 1 even more dependent on Li's result, mainly in order to localize where exactly the two results differ. Since Li's proof of his result is extremely difficult, however, I continue to work, now jointly with T. Fischer, on an independent and simpler proof of the theorem in Section 1.

I conclude this introduction with a diagram (Fig. 1) of the classes of subspaces encountered so far. The inclusions "Chebyshev $\subset$ continuous selection" and "$Z \subset$ almost Chebyshev" as well as the identity "unique continuous selection = continuous selection $\cap$ almost Chebyshev" have been explained already, and simple, at most 2-dimensional, examples of subspaces in positions 1 through 7 are easily constructed using the results in the present paper. Now, one glance at this diagram makes it evident why Z-subspaces should not be in the picture.

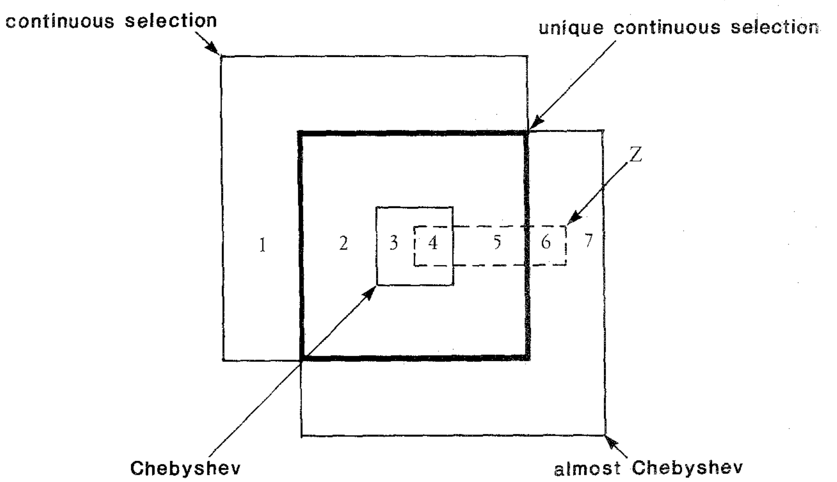

FIGURE 1

1. On the Approximating Subspaces of $C(X)$

Proposition. *The metric projection P of $C(X)$ onto G has a unique continuous selection iff it has a continuous selection and G is an almost Chebyshev subspace of $C(X)$.*

Proof. The "if" part follows directly from the definitions involved: If G is almost Chebyshev, then any two continuous selections for P coincide on a dense subset of $C(X)$, and therefore are identical.

We now turn to the "only if" part. Theorem and Lemma 3 of J. Blatter and L. Schumaker [6] combined yield immediately that if P has a continuous selection, then for every $f \in C(X)$ and for every $\varepsilon > 0$ there exists an $f_\varepsilon \in C(X)$ such that $\|f - f_\varepsilon\| < \varepsilon$ and

$$P(f_\varepsilon) = \{Sf\colon S \text{ a continuous selection for } P\}.$$

This implies immediately that if P has a unique continuous selection, then the set

$$U = \{f \in C(X)\colon P(f) \text{ is a singleton}\}$$

is dense in $C(X)$, and this, as A. L. Garkavi [14, pp. 171–172 of the English translation] has shown, is enough for G to be almost Chebyshev. For curiosity only, we include a much simpler proof of Garkavi's result for the present finite-dimensional case using the folkloric fact that in this case P is upper semi-continuous: For each $k \in \mathbb{N}$ set

$$U_k = \{f \in C(X)\colon P(f) \text{ is contained in some open subset of } G \text{ of diameter less than } 1/k\}.$$

Obviously, $U = \bigcap_{k \in \mathbb{N}} U_k$, and therefore, along with U, each U_k is dense in $C(X)$. Now observe that, by the very definition of upper semi-continuity, each U_k is open in $C(X)$.

A. L. Garkavi [15, Theorem I and last paragraph on p. 186 of the English translation] gave the following intrinsic description of almost Chebyshev subspaces of $C(X)$.

G is an almost Chebyshev subspace of $C(X)$ iff for any non-zero function g in G,

$$\operatorname{card} \operatorname{int} Z(g) \leqslant n - 1$$

(card = cardinal number of), and for any $k \leqslant n-1$ distinct isolated points $x_1, ..., x_k$ of X,

$$\dim\{g \in G\colon g(x_1) = \cdots = g(x_k) = 0\} \leqslant n - k.$$

We shall have occasion to use the following modification of Garkavi's conditions.

LEMMA. *G is an almost Chebyshev subspace of $C(X)$ iff for any non-zero function g in G,*

$$\text{card int } Z(g) \leqslant n - \dim\{h \in G: h = 0 \textit{ on } \text{int } Z(g)\}.$$

Proof. Suppose G satisfies Garkavi's conditions. If $g \in G \sim \{0\}$ is such that $\text{int } Z(g) \neq \varnothing$, then $\text{int } Z(g)$ consists of $k \leqslant n-1$ isolated points $x_1, ..., x_k$ of X, and therefore

$$\begin{aligned}&\dim\{h \in G: h = 0 \text{ on int } Z(g)\}\\ &\qquad = \dim\{h \in G: h(x_1) = \cdots = h(x_k) = 0\} \leqslant n-k.\end{aligned}$$

Thus G satisfies our condition.

Now suppose G satisfies our condition. If $g \in G \sim \{0\}$, then

$$\dim\{h \in G: h = 0 \text{ on int } Z(g)\} \geqslant 1,$$

and therefore $\text{card int } Z(g) \leqslant n-1$. Thus G satisfies the first of Garkavi's conditions. Now Garkavi's first condition obviously implies that his second condition holds for $k = n$, and therefore we may and shall prove the second condition by induction over $k = n, ..., 1$: Let $k \leqslant n-1$ distinct isolated points $x_1, ..., x_k$ of X be given and set

$$H = \{g \in G: g(x_1) = \cdots = g(x_k) = 0\}.$$

If X has no isolated points other than $x_1, ..., x_k$, then $\text{int } Z(g) = \{x_1, ..., x_k\}$ for any $g \in H \sim \{0\}$, and therefore

$$H = \{h \in G: h = 0 \text{ on int } Z(g)\}.$$

Thus $k \leqslant n - \dim H$ in this case. If X has an isolated point x_{k+1} distinct from $x_1, ..., x_k$, set

$$H' = \{h \in H: h(x_{k+1}) = 0\},$$

observe that $\dim H' \leqslant n-(k+1)$ by the hypothesis of our induction, and conclude that $\dim H \leqslant \dim H' + 1 \leqslant n-k$ also in this case. Thus G also satisfies the second of Garkavi's conditions.

F. Deutsch and G. Nürnberger [12] introduced weakly interpolating subspaces of $C(X)$.

G is called a *weakly interpolating subspace of* $C(X)$ if for any n distinct points $x_1, ..., x_n$ of X and any n signs $\sigma_1, ..., \sigma_n$ in $\{-1, 1\}$, there exists a non-zero function g in G such that, for each $i = 1, ..., n$, the function $\sigma_i g$ is non-negative on a neighborhood of x_i.

W. Li [18] introduces regular weakly interpolating subspaces of $C(X)$.

G is called a *regular weakly interpolating subspace of* $C(X)$ if for any non-empty finite subset $F = \{x_1, ..., x_k\}$ of X with the property that $F \sim \operatorname{int} Z(G(F)) \neq \varnothing$ ($G(F) = \{g \in G\colon g = 0$ on $F\}$ and $Z(G(F)) = \bigcap \{Z(g)\colon g \in G(F)\}$) and for any signs $\sigma_1, ..., \sigma_k$ in $\{-1, 1\}$, there exists a function g in G such that $F \sim \operatorname{int} Z(g) \neq \varnothing$ and, for each $i = 1, ..., k$, the function $\sigma_i g$ is non-negative on a neighborhood of x_i.

W. Li [18, Theorem 1.1] then gives the following intrinsic description of subspaces of $C(X)$ whose metric projection has a continuous selection.

The metric projection P of $C(X)$ onto G has a continuous selection iff G is a regular weakly interpolating subspace of $C(X)$.

And W. Li [18, Theorem 1.2] notes that for Z-subspaces of $C(X)$ this description can be much simplified.

If G is a Z-subspace of $C(X)$, then the metric projection P of $C(X)$ onto G has a continuous selection iff any non-zero function in G has at most n distinct zeros and G is a weakly interpolating subspace of $C(X)$.

THEOREM. *The metric projection P of $C(X)$ onto G has a unique continuous selection iff*

(1) *any non-zero function in G has at most n distinct zeros;*

(2) *for any $k \leqslant n$ distinct isolated points $x_1, ..., x_k$ of X, $\dim\{g \in G\colon g(x_1) = \cdots = g(x_k) = 0\} \leqslant n - k$; and*

(3) *G is a weakly interpolating subspace of $C(X)$.*

Proof. Suppose P has a unique continuous selection. By Li's first theorem, G is a regular weakly interpolating subspace of $C(X)$, and this is easily seen to imply (cf. W. Li [18, Lemma 4.1]) that G is a weakly interpolating subspace of $C(X)$, i.e., G satisfies condition (3). By the proposition, G is almost Chebyshev, and this, by Garkavi's description of almost Chebyshev subspaces, implies that G satisfies condition (2). Also, by the lemma,

$$\operatorname{card} \operatorname{int} Z(g) \leqslant n - \dim\{h \in G\colon h = 0 \text{ on } \operatorname{int} Z(g)\}$$

for all $g \in G \sim \{0\}$. Now, W. Li [18, Theorem 5.1] shows that whenever G is regular weakly interpolating, then

$$\operatorname{card} \operatorname{bdry} Z(g) \leqslant \dim\{h \in G\colon h = 0 \text{ on } \operatorname{int} Z(g)\}$$

(bdry = boundary of) for all $g \in G$. And these two inequalities combined obviously imply that G also satisfies condition (1).

Now suppose G satisfies conditions (1)–(3). By Garkavi's description of almost Chebyshev subspaces, conditions (1) and (2) together imply that G is almost Chebyshev, and therefore, by the proposition, we need only show that P has a continuous selection. This we do by showing that conditions (1) and (3) together imply that G is regular weakly interpolating (and then, of course, appealing to the other half of Li's first theorem): Let $F = \{x_1, ..., x_k\}$ be a non-empty finite subset of X with the property that $F \sim \operatorname{int} Z(G(F)) \neq \emptyset$ and let $\sigma_1, ..., \sigma_k \in \{-1, 1\}$. If $k > n$, then, by condition (1), $G(F) = \{0\}$ whence $Z(G(F)) = X$ and therefore $F \subset \operatorname{int} Z(G(F))$, a contradiction. Thus $k \leqslant n$ and therefore, by condition (3), there exists $g \in G \sim \{0\}$ such that, for each $i = 1, ..., k$, $\sigma_i g \geqslant 0$ on a neighborhood of x_i. Now, if all points of F are isolated points of X, then F is open and therefore ($F \subset Z(G(F))$!) $F \subset \operatorname{int} Z(G(F))$, the same contradiction. Thus not all points of F are isolated points of X and therefore, again by condition (1) (note that $g \neq 0$!), $F \sim \operatorname{int} Z(g) \neq \emptyset$. This does it.

Remarks. 1. Simple 1-dimensional examples show that no two of the conditions (1)–(3) imply the third.

2. Our proof that conditions (1) and (3) together imply that G is regular weakly interpolating is a variation of Li's argument to deduce the "if" part of his second theorem from that of the first.

3. The special cases of the theorem that G is 1-dimensional and X arbitrary and that G is arbitrary and X a real interval were proved by J. Blatter and L. Schumaker [6, 7] building on earlier work of A. J. Lazar, D. E. Wulbert, and P. D. Morris [17] and G. Nürnberger and M. Sommer (see [20] and the references therein).

2. On the Underlying Spaces X

W. Li [18, Theorem 6.1] proves the following theorem on determinants.

If G is a regular weakly interpolating subspace of $C(X)$, then, given a basis $g_1, ..., g_n$ for G and given distinct points $x_1, ..., x_n$ of X, there exist neighborhoods U_i of the x_i and a sign σ in $\{-1, 1\}$ such that for any points y_i in the U_i,

$$\sigma \det \begin{bmatrix} g_1(y_1) \cdots g_n(y_1) \\ \vdots \qquad \vdots \\ g_1(y_n) \cdots g_n(y_n) \end{bmatrix} \geqslant 0.$$

And A. L. Garkavi [15, p. 186 of the English translation] stated without proof the following theorem on determinants.

If G is an almost Chebyshev subspace of $C(X)$, then, given a basis $g_1, \ldots, g_n$ for G, given distinct points $x_1, \ldots, x_n$ of X, and given neighborhoods U_i of the x_i, there exist points y_i in the U_i such that

$$\det \begin{bmatrix} g_1(y_1) \cdots g_n(y_1) \\ \vdots \qquad \vdots \\ g_1(y_n) \cdots g_n(y_n) \end{bmatrix} \neq 0.$$

Since Garkavi actually stated a false version of this theorem, we include a

Proof. The proof is by induction over the dimension n of G and uses Garkavi's description of almost Chebyshev subspaces of $C(X)$.

The theorem is trivially true for $n=1$. Suppose then that the theorem has been proved for $n \leqslant m$ and that $\dim G = m+1$. Let $g_1, \ldots, g_{m+1}$ be a basis for G, let $x_1, \ldots, x_{m+1}$ be distinct points of X, and, for each i, let U_i be a neighborhood of x_i. We distinguish two cases.

Suppose X has no isolated point. In this case, let z be any point in X which is not a common zero of the functions in G. We may and shall assume that z is not one of $x_1, \ldots, x_{m+1}$. Set $H = \{g \in G\colon g(z) = 0\}$ and observe that H is an m-dimensional almost Chebyshev subspace of $C(X)$. Let $h_1, \ldots, h_m$ be a basis for H and let $h_{m+1} \in G$ be linearly independent of $h_1, \ldots, h_m$. By hypothesis, there exist points $y_1, \ldots, y_m$ in $U_1, \ldots, U_m$, respectively, such that

$$\det(h_i(y_j))_{i,j=1,\ldots,m} \neq 0.$$

Set

$$g(x) = \det \begin{bmatrix} h_1(y_1) \cdots h_m(y_1) & h_{m+1}(y_1) \\ \vdots \qquad \vdots & \vdots \\ h_1(y_m) \cdots h_m(y_m) & h_{m+1}(y_m) \\ h_1(x) \cdots h_m(x) & h_{m+1}(x) \end{bmatrix}, \qquad x \in X.$$

g is a non-zero function in G, and therefore $g(y_{m+1}) \neq 0$ for some $y_{m+1} \in U_{m+1}$. Now observe that $\det(g_i(y_j))_{i,j=1,\ldots,m+1}$ is a non-zero multiple of $g(y_{m+1})$.

Now suppose X has isolated points. If all of $x_1, \ldots, x_{m+1}$ are isolated points of X, then $\det(g_i(x_j))_{i,j=1,\ldots,m+1} \neq 0$ and we are done. Suppose therefore in addition that not all of $x_1, \ldots, x_{m+1}$ are isolated points of X. In this case, let z be an arbitrary isolated point of X. We may and shall assume that z is not one of $x_1, \ldots, x_m$ and that either $x_{m+1} = z$ or else x_{m+1} is not an isolated point of X. Set $H = \{g \mid X \sim \{z\}\colon g \in G \text{ and } g(z) = 0\}$ ($\mid$ = restricted to), and observe that H is an m-dimensional almost Chebyshev subspace of $C(X \sim \{z\})$. Let $h_1, \ldots, h_m \in G$ be such that their restrictions to $X \sim \{z\}$ are a basis for H, and let $h_{m+1} \in G$ be linearly independent of $h_1, ..., h_m$. Again by hypothesis, there exist points $y_1, ..., y_m$ in $U_1, ..., U_m$, respectively, such that

$$\det(h_i(y_j))_{i,j=1,...,m} \neq 0.$$

As before, set

$$g(x) = \det \begin{bmatrix} h_1(y_1) \cdots h_m(y_1) & h_{m+1}(y_1) \\ \vdots \quad\quad \vdots & \vdots \\ h_1(y_m) \cdots h_m(y_m) & h_{m+1}(y_m) \\ h_1(x) \cdots h_m(x) & h_{m+1}(x) \end{bmatrix}, \qquad x \in X.$$

Then, as before, g is a non-zero function in G, and therefore, as before, $g(y_{m+1}) \neq 0$ for some $y_{m+1} \in U_{m+1}$ if x_{m+1} is not on isolated point of X; if x_{m+1} is an isolated point of X, however, then $x_{m+1} = z$, and therefore

$$g(x_{m+1}) = h_{m+1}(z) \det(h_i(y_j))_{i,j=1,...,m} \neq 0.$$

The same argument as in the first case now concludes the proof.

LEMMA. *If the metric projection P of $C(X)$ onto G has a unique continuous selection and if the dimension n of G is $\geqslant 2$, then*

(i) *if X is homeomorphic to S^1, n is odd;*

(ii) *no proper subspace of X is homeomorphic to S^1; and*

(iii) *no subspace of X is homeomorphic to the subspace*

$$\perp = \{(r, s) \in \mathbb{R}^2\colon -1 \leqslant r \leqslant 1, 0 \leqslant s \leqslant 1 \text{ and } rs = 0\}$$

of $\mathbb{R}^2$.

Proof. (i) We suppose S^1 has been identified with X and we assume to the contrary that n is even. Let $g_1, ..., g_n$ be a basis for G. By Garkavi's theorem on determinants, there exist distinct points $x_{1,0}, ..., x_{n,0}$ of S^1 in positive order such that

$$\det(g_i(x_{j,0}))_{i,j=1,...,n} \neq 0.$$

Set

$$\Delta = \{p = (x_1, ..., x_n) \in (S^1)^n\colon \text{two of } x_1, ..., x_n \text{ coincide}\},$$

denote by C the connected component of $p_0 = (x_{1,0}, ..., x_{n,0})$ in $(S^1)^n \sim \Delta$, and define $\delta\colon C \to \mathbb{R}$ by

$$\delta(p) = \det(g_i(x_j))_{i,j=1,...,n}, \qquad p = (x_1, ..., x_n) \in C.$$

It is easily seen that $p_1=(x_{2,0}, ..., x_{n,0}, x_{1,0})$ belongs to the path-component of p_0 in $(S^1)^n \sim \Delta$, and therefore to C. Since n is even, $\delta(p_1)=-\delta(p_0)$. Thus $\text{pos}(\delta)=\{p\in C: \delta(p)>0\}$ and $\text{neg}(\delta)=\{p\in C: \delta(p)<0\}$ are both non-empty. Since δ is continuous and C is connected, $Z(\delta)$ is also non-empty. Now, it is easily seen that C is open in $(S^1)^n$, and therefore, by Garkavi's theorem on determinants, $\text{int}_C Z(\delta)=\varnothing$ whence $\text{cl}_C \text{pos}(\delta) \cup \text{cl}_C \text{neg}(\delta)=C$ (cl_C = closure in C of). Since C is connected, $\text{cl}_C \text{pos}(\delta) \cap \text{cl}_C \text{neg}(\delta) \neq \varnothing$, a contradiction to Li's theorem on determinants.

(ii) We assume to the contrary that S^1 is homeomorphic to a proper subspace of X and we suppose S^1 has been identified with this subspace. We distinguish two cases.

Suppose n is even. In this case, $\{g|S^1: g\in G\}$ is an n-dimensional subspace of $C(S^1)$ which satisfies conditions (1)–(3) of the theorem in Section 1 (for the dimension and for (3) note that no non-zero function in G is zero on S^1, and for (2) note that S^1 has no isolated points), i.e., is an even-dimensional subspace of $C(S^1)$ whose metric projection has a unique continuous selection, a contradiction to (i).

Now suppose n is odd. Since S^1 is a proper subset of X, there is a point z in $X \sim S^1$ which is not a common zero of the functions in G (there is at most one common zero of the functions in G, and if there is one, it is not an isolated point of X). Then $H=\{g|S^1: g\in G \text{ and } g(z)=0\}$ is an $(n-1)$-dimensional subspace of $C(S^1)$ which satisfies the conditions (1)–(3) (for $n-1$!) of the theorem in Section 1 (for (3) argue as follows: Let $x_1, ..., x_{n-1}$ be distinct points of S^1 and let $\sigma_1, ..., \sigma_{n-1}$ be signs in $\{-1, 1\}$. There exist non-zero functions g^+ and g^- in G such that, for each $i=1, ..., n-1$, $\sigma_i g^+$ and $\sigma_i g^-$ are non-negative on a neighborhood of x_i and such that g^+ and $-g^-$ are non-negative on a neighborhood of z. Then, obviously, some convex combination of g^+ and g^- belongs to H and ...). Thus H is an even-dimensional subspace of $C(S^1)$ whose metric projection has a unique continuous selection, again a contradiction to (i).

(iii) We assume to the contrary that $\perp$ is homeomorphic to a subspace of X and we suppose $\perp$ has been identified with this subspace. As before, $H=\{g|\perp: g\in G\}$ is an n-dimensional subspace of $C(\perp)$ whose metric projection has a unique continuous selection. Let $h_1, ..., h_n$ be a basis for H. By Garkavi's theorem on determinants, there exist distinct points $x_{1,0}, ..., x_{n,0}$ of the vertical branch of $\perp$ in ascending order such that

$$\det(h_i(x_{j,0}))_{i,j=1,...,n} \neq 0.$$

Set

$$\Delta=\{p=(x_1, ..., x_n)\in \perp^n: \text{two of } x_1, ..., x_n \text{ coincide}\},$$

denote by C the connected component of $p_0 = (x_{1,0}, ..., x_{n,0})$ in $\perp^n \sim \Delta$, and define $\delta\colon C \to \mathbb{R}$ by

$$\delta(p) = \det(h_i(x_j))_{i,j=1,...,n}, \qquad p = (x_1, ..., x_n) \in C.$$

A cute "four movements" argument shows that $p_1 = (x_{2,0}, x_{1,0}, x_{3,0}, ..., x_{n,0})$ belongs to the path-component of p_0 in $\perp^n \sim \Delta$, and therefore to C. Obviously, $\delta(p_1) = -\delta(p_0)$, and now a contradiction is reached just like in the proof of (i).

In our extension of Mairhuber's theorem we shall have occasion to distinguish grades of non-Chebyshev: For an integer $0 \leqslant k \leqslant n-1$, G is called a *k-Chebyshev subspace of* $C(X)$ if for any f in $C(X)$ the dimension of the set $P(f)$ of best approximations of f in G is at most k; in particular, G is a 0-Chebyshev subspace of $C(X)$ iff it is a Chebyshev subspace. G. S. Rubinstein [21] extended Haar's theorem as follows.

G is a k-Chebyshev subspace of $C(X)$ iff for any $n-k$ distinct points $x_1, ..., x_{n-k}$ of X,

$$\dim\{g \in G\colon g(x_1) = \cdots = g(x_{n-k}) = 0\} \leqslant k.$$

THEOREM. *The set of integers $n \geqslant 1$ with the property that $C(X)$ contains an n-dimensional vector subspace whose metric projection has a unique continuous selection is*

(i) $\{1, ..., k\}$ *if X has only a finite number k of points, and in this case all examples are necessarily Chebyshev;*

(ii) $\{1, 2, 3, ...\}$ *if X is homeomorphic to an infinite closed subspace of the unit interval* $[0, 1]$, *and in this case for all $n > 1$ there are examples which are Chebyshev and examples which are not $(n-1)$-Chebyshev;*

(iii) $\{1, 3, 5, ...\}$ *if X is homeomorphic to the unit sphere S^1, and in this case for all odd $n > 1$ there are examples which are Chebyshev and examples which are not $(n-1)$-Chebyshev;*

(iv) $\{1, 2\}$ *if X is homeomorphic to a closed subspace of some interval with split points $T(\varnothing, D^+)$, $D^+ \subset [0, 1)$, and if the set of points t in D^+ with the property that both t and t^+ are in that subspace is uncountable, and in this case all 2-dimensional examples are necessarily 1-Chebyshev but not Chebyshev; and*

(v) $\{1\}$ *in all other cases.*

Proof. The proof of the theorem is divided into three parts, namely an embedding theorem, an analysis of metrizable and non-metrizable closed subspaces of intervals with split points, and the construction of examples. We begin with the

EMBEDDING THEOREM. *If for some integer $n \geqslant 2$, $C(X)$ contains an n-dimensional vector subspace G whose metric projection has a unique continuous selection, then either X is homeomorphic to a subspace of some interval with split points $T(\varnothing, D^+)$, $D^+ \subset [0, 1)$, or else X is homeomorphic to the unit sphere S^1 (and n is odd).*

Proof. We prove this theorem by induction over the dimension n of G. For $n = 2$ we distinguish two cases.

Suppose $n = 2$ and G is 1-Chebyshev. Fix a basis g_1, g_2 for G. By Rubinstein's theorem, g_1 and g_2 have no common zero, and therefore we may define $\varphi: X \to S^1$ by

$$\varphi(x) = \left(\frac{g_1(x)}{(g_1^2(x) + g_2^2(x))^{1/2}}, \frac{g_2(x)}{(g_1^2(x) + g_2^2(x))^{1/2}} \right), \qquad x \in X.$$

φ is obviously continuous. Set

$$L_p = \{(u, v) \in \mathbb{R}^2 : su - rv = 0\}, \qquad p = (r, s) \in S^1,$$

i.e., L_p is the line in $\mathbb{R}^2$ through the origin and p. It is clear then that

$$\varphi(x) \in L_p \text{ iff } x \text{ is a zero of } sg_1 - rg_2 \in G \sim \{0\},$$
$$x \in X, \text{ and } p = (r, s) \in S^1.$$

Since $\operatorname{card} Z(g) \leqslant 2$ for all $g \in G \sim \{0\}$, $\operatorname{card} \varphi^{-1}[L_p] \leqslant 2$ for all $p \in S^1$. This shows that φ is not surjective: Were φ surjective it would necessarily be injective too, and therefore a homeomorphism of X onto S^1, a contradiction to item (i) of the lemma. Thus $\varphi[X]$ is a proper closed subset of S^1, and therefore contained in some open arc. This arc is homeomorphic to the open unit interval $(0, 1)$, and we set $\psi = \eta \circ \varphi$, where η is some such homeomorphism. We have already seen that $\operatorname{card} \psi^{-1}[\{t\}] \leqslant 2$ for all $t \in (0, 1)$. Suppose now t is a point in $(0, 1)$ with the property that $\psi^{-1}[\{t\}]$ contains two points, say x_1 and x_2. Then with $p = (r, s) \in S^1$ such that $\eta(p) = t$ we have $\varphi^{-1}[\{p\}] = \{x_1, x_2\}$. This is to say that $g(x_1) = cg(x_2)$ for all $g \in G$, where

$$c = \frac{(g_1^2(x_1) + g_2^2(x_1))^{1/2}}{(g_1^2(x_2) + g_2^2(x_2))^{1/2}} > 0.$$

Set

$$g(x) = \det \begin{bmatrix} g_1(x) & g_2(x) \\ g_1(x_2) & g_2(x_2) \end{bmatrix}, \qquad x \in X.$$

Since g_1 and g_2 have no common zeros, $g \in G \sim \{0\}$. Also

$$g(x) = \det \begin{bmatrix} g_1(x) & g_2(x) \\ \dfrac{g_1(x_1)}{c} & \dfrac{g_2(x_1)}{c} \end{bmatrix} = -\frac{1}{c} \det \begin{bmatrix} g_1(x_1) & g_2(x_1) \\ g_1(x) & g_2(x) \end{bmatrix}, \qquad x \in X.$$

By Li's theorem on determinants, there exist disjoint neighborhoods U_1 and U_2 of x_1 and x_2, respectively, and a sign $\sigma \in \{-1, 1\}$ such that

$$\sigma \det \begin{bmatrix} g_1(y_1) & g_2(y_1) \\ g_1(y_2) & g_2(y_2) \end{bmatrix} \geqslant 0 \qquad \text{for any} \quad y_1 \in U_1 \text{ and } y_2 \in U_2,$$

and applying this to the two representations of g, we see immediately that $\sigma g \geqslant 0$ on U_1 and $-\sigma c g \geqslant 0$ on U_2. Now, since x_1 and x_2 are zeros of both g and $sg_1 - rg_2$, and since no non-zero function in G has more than two zeros, g and $sg_1 - rg_2$ must be multiples of each other, and we conclude (note that $c > 0$) that for some sign $\sigma' \in \{-1, 1\}$, $\sigma'(sg_1 - rg_2) > 0$ on $U_1 \sim \{x_1\}$ and $\sigma'(sg_1 - rg_2) < 0$ on $U_2 \sim \{x_2\}$, and this, in the geometric language introduced above, means that $\varphi[U_1 \sim \{x_1\}]$ and $\varphi[U_2 \sim \{x_2\}]$ lie in opposite of the two open half-planes determined by the line L_p, or (note that the homeomorphism η is necessarily "monotone") that $\psi[U_1 \sim \{x_1\}]$ and $\psi[U_2 \sim \{x_2\}]$ lie in opposite of the two open intervals $(0, t)$ and $(t, 1)$. We finally note that, by Garkavi's description of almost Chebyshev subspaces, not both of x_1 and x_2 can be isolated points of X, i.e., at least one of $U_1 \sim \{x_2\}$ and $U_2 \sim \{x_2\}$ is non-empty. We set

$$D^+ = \{t \in (0, 1) \colon \operatorname{card} \psi^{-1}[\{t\}] = 2\}$$

and define $\psi^+ \colon X \to T(\varnothing, D^+)$ by

if $t \in \psi[X]$ is such that $\psi^{-1}[\{t\}] = \{x\}$, then $\psi^+(x) = \psi(x)$; and

if $t \in \psi[X]$ is such that $\psi^{-1}[\{t\}] = \{x_1, x_2\}$ with, say, x_1 not an isolated point of X, then $\psi^+(x_1) = t$ and $\psi^+(x_2) = t^+$ or $\psi^+(x_1) = t^+$ and $\psi^+(x_2) = t$ according as ψ maps some neighborhood of x_1 into $(0, t]$ or $[t, 1)$.

ψ^+ is well-defined and injective. That ψ^+ is also continuous is all but a direct consequence of the definitions involved. Since X is compact, ψ^+ is a homeomorphism onto $\psi^+[X] \subset T(\varnothing, D^+)$, and we are done in this case.

Next suppose $n = 2$ and G is not 1-Chebyshev. Fix a basis g_1, g_2 for G. By Rubinstein's theorem, g_1 and g_2 have a common zero z. By our general assumption on G, z is the only common zero of g_1 and g_2, any $g \in G \sim \{0\}$

has at most one zero in $X\sim\{z\}$, and z is not an isolated point of X. Therefore we may define $\varphi\colon X\sim\{z\}\to S^1$ by

$$\varphi(x)=\left(\frac{g_1(x)}{(g_1^2(x)+g_2^2(x))^{1/2}},\frac{g_2(x)}{(g_1^2(x)+g_2^2(x))^{1/2}}\right),\qquad x\in X\sim\{z\},$$

and φ, aside from being continuous, has the property that card $\varphi^{-1}[L_p]\leqslant 1$ for all $p\in S^1$, where the L_p's are the lines defined in the first part of the proof, and this is to say that φ is injective and $\varphi[X\sim\{z\}]$ contains no pair of antipodal points of S^1; moreover, for every $x\in X\sim\{z\}$, there exists a neighborhood U_x of z which does not contain x such that φ maps the non-empty set $U_x\sim\{z\}$ into one of the two open half-planes determined by the line $L_{\varphi(x)}$: Let $x\in X\sim\{z\}$ and set

$$g(y)=\det\begin{bmatrix} g_1(y) & g_2(y)\\ g_1(x) & g_2(x)\end{bmatrix},\qquad y\in X.$$

Then $g\in G\sim\{0\}$ and, by Li's theorem on determinants, there exist a neighborhood U_x of z which does not contain x and a sign $\sigma\in\{-1,1\}$ such that $\sigma g\geqslant 0$ on U_x. Now observe that $\sigma g=\sigma(g_2(x)\,g_1-g_1(x)\,g_2)>0$ on $U_x\sim\{z\}$ and that

$$L_{\varphi(x)}=\{(u,v)\in\mathbb{R}^2\colon (g_1^2(x)+g_2^2(x))^{-1/2}\,(g_2(x)\,u-g_1(x)v)=0\}.$$

Let Σ be the set of points p in S^1 with the property that for some net $\{x_i\}_{i\in I}$ in $X\sim\{z\}$ which converges to z, the net $\{\varphi(x_i)\}_{i\in I}$ converges to p. Since z is not an isolated point of X, Σ is not empty. Given $x\in X\sim\{z\}$, then, since φ maps $U_x\sim\{z\}$ into one of the two open half-planes determined by the line $L_{\varphi(x)}$, Σ is contained in one of the two closed half-planes determined by $L_{\varphi(x)}$. Since $X\sim\{z\}$ contains at least two distinct points x_1 and x_2, and since the lines $L_{\varphi(x_1)}$ and $L_{\varphi(x_2)}$ do not coincide, it follows that Σ is contained in one of the four closed quadrants determined by $L_{\varphi(x_1)}$ and $L_{\varphi(x_2)}$; in particular, Σ contains no pair of antipodal points of S^1. Now, would Σ contain three points, say p_1, p_2, and p_3 with p_2 on the minor arc between p_1 and p_3, then, since p_2 is the limit of some net $\{\varphi(x_i)\}_{i\in I}$, p_1 and p_3 would eventually be contained in opposite of the two open half-planes determined by the line $L_{\varphi(x_i)}$, a contradiction. Thus Σ has either one or two points (and both can actually occur!). We treat these two cases separately.

Suppose $\Sigma=\{p,q\}$. In this case the situation is as shown in Fig. 2, where A and B denote the minor closed arcs between p and $-q$ and between q and $-p$, respectively. Since for any $x\in X\sim\{z\}$, p and q belong to the same of the two closed half-planes determined by the line $L_{\varphi(x)}$, $\varphi[X\sim\{z\}]\subset A\mathbin{\dot\cup} B$ ($\dot\cup=$disjoint union of). This implies that

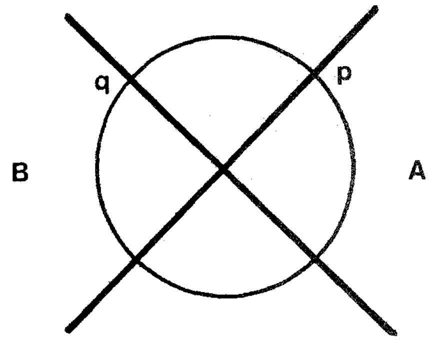

FIGURE 2

neither p nor q belongs to $\varphi[X \sim \{z\}]$: Were $p = \varphi(x)$ for some $x \in X \sim \{z\}$, for example, then $\varphi[U_x \sim \{z\}]$, being contained in one of the two open half-planes determined by the line L_p and containing p in its closure, would have to be contained in A, and therefore could not contain q in its closure, a contradiction. Now, let $A \cup B/p = q$ be the quotient space of $A \cup B$ obtained by identifying the points p and q, let π be the quotient map, and define $\psi: X \to A \cup B/p = q$ by

$$\psi(x) = \begin{cases} \pi(\varphi(x)) & \text{if } x \in X \sim \{z\} \\ \pi(p) = \pi(q) & \text{if } x = z. \end{cases}$$

ψ is continuous and injective, and therefore a homeomorphism of X onto $\psi[X]$. Obviously, $A \cup B/p = q$ is homeomorphic to the unit interval $[0, 1]$. So much for this case.

Now suppose $\Sigma = \{p\}$. In this case, define $\tilde{\varphi}: X \to S^1$ by

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x \in X \sim \{z\} \\ p & \text{if } x = z. \end{cases}$$

$\tilde{\varphi}$ is a continuous extension of φ to all of X. If $p \notin \varphi[X \sim \{z\}]$, $\tilde{\varphi}$ is also injective, and therefore a homeomorphism of X onto the proper closed subset $\tilde{\varphi}[X]$ of S^1. Thus X can be embedded into the unit interval $[0, 1]$ in this case, and we are left with the case that $p = \varphi(y)$ for some $y \in X \sim \{z\}$ (this can actually occur!). The situation then is as shown in Fig. 3, where A denotes that of the two closed semi-circles between p and $-p$ which contains $\varphi[U_y \sim \{z\}]$ in its interior. If C is an open and closed subset of X which contains y but not z, then $\tilde{\varphi} | C$ and $\tilde{\varphi} | X \sim C$ are homeomorphisms onto proper closed subsets of S^1 which induce in the obvious fashion an embedding of X into the unit interval $[0, 1]$. We prove now that such a C indeed exists: Suppose not. Let V be an arbitrary open

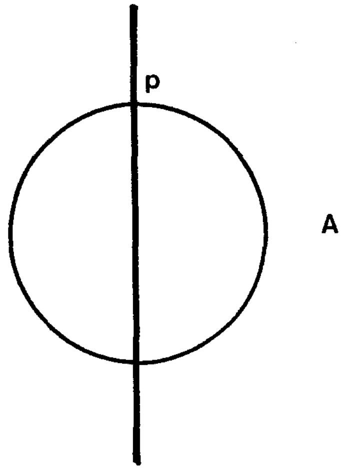

FIGURE 3

neighborhood of y whose closure does not contain z. The set $\tilde{\varphi}^{-1}[A] \cap (X \sim V)$ is a closed neighborhood of z which does not contain y, and the identity

$$\tilde{\varphi}^{-1}[A] \cap (X \sim V) = \{z\} \cup (\tilde{\varphi}^{-1}[\text{int } A] \cap (X \sim V))$$

shows that $\tilde{\varphi}^{-1}[A] \cap (X \sim V)$ is also open if $\tilde{\varphi}^{-1}[\text{int } A] \subset X \sim V$. Thus, by our supposition, there exists a net $\{y_i\}_{i \in I}$ in X which converges to y such that $\tilde{\varphi}(y_i) \in \text{int } A$ for all i. Now, let N be a closed neighborhood of z which does not contain y. By our supposition, N is not open, and therefore $\tilde{\varphi}[\text{bdry } N]$ (bdry = boundary of) is a non-empty closed subset of S^1 which does not contain p. Thus, denoting by B_i the minor closed arc between p and $\tilde{\varphi}(y_i)$, we have that eventually $y_i \in X \sim N$ and $\tilde{\varphi}^{-1}[B_i] \cap \text{bdry } N = \emptyset$. For such an i, $\tilde{\varphi}^{-1}[B_i] \cap N$ is a closed neighborhood of z which does not contain y and the identity

$$\begin{aligned} \tilde{\varphi}^{-1}[B_i] \cap N &= (\tilde{\varphi}^{-1}[B_i] \cap \text{int } N) \cup (\tilde{\varphi}^{-1}[B_i] \cap \text{bdry } N) \\ &= \tilde{\varphi}^{-1}[B_i] \cap \text{int } N = \{z\} \cup (\tilde{\varphi}^{-1}[\text{int } B_i] \cap \text{int } N) \end{aligned}$$

shows that $\tilde{\varphi}^{-1}[B_i] \cap N$ is also open, a contradiction.

With the case $n = 2$ now out of the way, suppose that the embedding theorem has been proved for $2 \leqslant n \leqslant m$ and that $\dim G = m + 1$. We distinguish three cases.

Suppose X has an isolated point z. In this case, $H = \{g \mid X \sim \{z\} : g \in G$ and $g(z) = 0\}$ is an m-dimensional vector subspace of $C(X \sim \{z\})$ whose metric projection has a unique continuous selection. By item (ii) of the lemma, $X \sim \{z\}$ is not homeomorphic to S^1, and therefore, by hypothesis, $X \sim \{z\}$ can be embedded into some interval with split points. It is clear then how X can be so embedded.

Now suppose X has no isolated point but is not connected. Let X be the disjoint union $A \cup B$ of two non-empty closed subsets A and B. Since A is open, it has no isolated point. Since B is not a singleton, it contains a point z which is not a common zero of the functions in G. Thus, $H = \{g \mid A: g \in G$ and $g(z) = 0\}$ is an m-dimensional vector subspace of $C(A)$ whose metric projection has a unique continuous selection. By item (ii) of the lemma, A is not homeomorphic to S^1, and therefore, by hypothesis, A can be embedded into some interval with split points. By symmetry, the same holds for B, and thus for X.

Finally, suppose X is connected. Since X has at least three points, G contains a function g which has positive values, negative values, and zeros. Since $Z(g)$ is finite and since $X \sim Z(g)$ is no longer connected, there exist a finite, possibly empty, set F of zeros of g and another zero y of g such that $X \sim F$ is still connected but $(X \sim F) \sim \{y\}$ is not. Let $(X \sim F) \sim \{y\}$ be the disjoint union $A \cup B$ of two non-empty closed subsets A and B. It is easily seen that the closures of A and B in $X \sim F$ are the sets $A \cup \{y\}$ and $B \cup \{y\}$, and that these sets are connected. It follows that $\mathrm{cl}\, A$ and $\mathrm{cl}\, B$ (closures in X!) are also connected, and it is clear that $\mathrm{cl}\, A \cup \mathrm{cl}\, B = X$. Since B is open in X, it contains a point z which is not a common zero of the functions in G. Since $z \in B, z \notin \mathrm{cl}\, A$. Thus, $H = \{g \mid \mathrm{cl}\, A: g \in G$ and $g(z) = 0\}$ is an m-dimensional vector subspace of $C(\mathrm{cl}\, A)$ whose metric projection has a unique continuous selection. By item (ii) of the lemma, $\mathrm{cl}\, A$ is not homeomorphic to S^1, and therefore by hypothesis, there exists a homeomorphism η of $\mathrm{cl}\, A$ onto a subspace of some interval with split points $T(\emptyset, D^+), D^+ \subset [0, 1)$. $\eta[\mathrm{cl}\, A]$ is a closed connected subset of $T(\emptyset, D^+)$ with more than one point. As in any compact totally ordered topological space (see the Appendix), this means that $\eta[\mathrm{cl}\, A]$ is the closed interval (in $T(\emptyset, D^+)$!) $[\inf \eta[\mathrm{cl}\, A], \sup \eta[\mathrm{cl}\, A]]$ (inf (sup) = infimum (supremum) of), that $\eta[\mathrm{cl}\, A]$ contains no gaps, and that $\inf \eta[\mathrm{cl}\, A] < \sup \eta[\mathrm{cl}\, A]$. The "no gaps" condition in the case at hand means that for no $t \in D^+$ can $\eta[\mathrm{cl}\, A]$ contain both t and t^+. Thus, the restriction to $\eta[\mathrm{cl}\, A]$ of the canonical projection π of $T(\emptyset, D^+)$ onto the unit interval $[0, 1]$ is (continuous and) injective, and $\varphi = \pi \circ \eta$ is a homeomorphism of $\mathrm{cl}\, A$ onto a non-degenerate closed subinterval of $[0, 1]$; and we may and shall assume that this subinterval is all of $[0, 1]$. Since $A \cup \{y\}$ is connected and dense in $\mathrm{cl}\, A$, $\varphi[A \cup \{y\}]$ is one of $[0, 1]$, $[0, 1)$, $(0, 1]$, and $(0, 1)$. It follows that $\varphi[\mathrm{bdry}\, A \sim \{y\}]$ is contained in $\{0, 1\}$. The point $\varphi(y)$ may or may not be one of 0 and 1. By symmetry, there exists a homeomorphism ψ of $\mathrm{cl}\, B$ onto $[0, 1]$ such that $\psi[\mathrm{bdry}\, B \sim \{y\}]$ is contained in $\{0, 1\}$, and again the point $\psi(y)$ may or may not be one of 0 and

1. It is obvious now that X is either homeomorphic to one of the three plane figures shown in Fig. 4, where the circle marks the image of the point y, or else to a figure obtained from one of these by identifying some of its extremities. The first figure itself is homeomorphic to the unit interval $[0, 1]$, and the figure obtained from it by identifying its two extremities is homeomorphic to the unit sphere S^1; the other two figures, however, as well as any figure obtained from one of them by identifying some of its extremities, all contain the figure $\perp$ of item (iii) of the lemma, and therefore the possibility that X is homeomorphic to one of these is excluded. The embedding theorem is proved.

We now turn to the second part of the proof of the theorem. We suppose that for some integer $n \geqslant 2$, $C(X)$ contains an n-dimensional vector subspace G whose metric projection has a unique continuous selection and that X is not homeomorphic to the unit sphere S^1. Then, by the embedding theorem, X is homeomorphic to a subspace of some interval with split points $T(\emptyset, D^+)$, $D^+ \subset [0, 1)$, and we assume X has been identified with this subspace. Let Σ be the set of all points t in D^+ which have the property that both t and t^+ are in X. We distinguish two cases.

Suppose Σ is countable. There exists a function φ: $[0, 1] \to \mathbb{R}$ which is strictly increasing from $\varphi(0) = 0$ to $\varphi(1) = 1$, and which is left-continuous at all points of $(0, 1]$ and right-continuous precisely at the points of $[0, 1) \sim \Sigma$: This is obvious if Σ is finite, and if Σ is not finite, say $\Sigma = \{t_1, t_2, ...\}$, an example of such a φ is

$$\varphi(t) = \frac{1}{2} t + \Sigma_{t_i < t} \frac{1}{2^{i+1}}, \qquad t \in [0, 1],$$

where the empty sum is taken to be zero. As we shall see in the Appendix, there exists a unique non-decreasing and continuous function $\tilde{\varphi}$ on $T(\emptyset, D^+)$ such that $\tilde{\varphi} \mid [0, 1] = \varphi$, and it is all but obvious that $\tilde{\varphi} \mid X$ is strictly increasing. Thus, $\tilde{\varphi} \mid X$ is an embedding of X into the unit interval $[0, 1]$; in particular, X is metrizable.

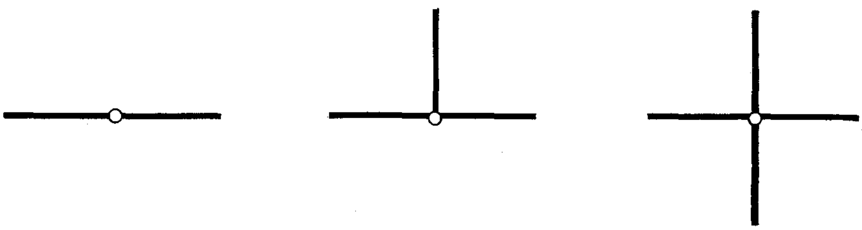

FIGURE 4

Now suppose Σ is not countable. For each $t \in \Sigma$, define $f_t: X \to \mathbb{R}$ by

$$f_t(x) = \begin{cases} 1 & \text{if } x \leqslant t, \\ 0 & \text{if } x \geqslant t^+, \end{cases} \quad x \in X.$$

Clearly, all the f_t's are continuous, and the distance between any two of them is 1. Thus, $C(X)$ is not separable, and this is well known to mean just that X is not metrizable. Next, fix a basis $g_1, ..., g_n$ for G. Clearly, each of the g_i's has a continuous extension $\tilde{\psi}_i$ to all of $T(\varnothing, D^+)$. As we shall see in the Appendix, the functions $\psi_i = \tilde{\psi}_i \mid [0, 1]$ are regulated functions on $[0, 1]$ which are left-continuous at all points of $(0, 1]$ and right-continuous at all points of $[0, 1) \sim D^+$. By the classical fact that the set of discontinuities of a regulated real-valued function on $[0, 1]$ is countable, for each i there exists a countable subset D_i of D^+ such that ψ_i is also right-continuous at all points of $D^+ \sim D_i$. It is all but obvious that a ψ_i is right-continuous at a point t in D^+ iff $\tilde{\psi}_i(t) = \tilde{\psi}_i(t^+)$. Thus, if we set $D = \bigcup_{i=1}^n D_i$, then $g(t) = g(t^+)$ for all $g \in G$ and all $t \in \Sigma \sim D$. Now, since D is countable and Σ is not, $\Sigma \sim D$ is uncountable and therefore contains $n-1$ distinct points $t_1, ..., t_{n-1}$. Clearly, there exists a non-zero function g in G such that $g(t_1) = \cdots = g(t_{n-1}) = 0$. Then, as we have just seen, also $g(t_1^+) = \cdots = g(t_{n-1}^+) = 0$, so that g has $2(n-1)$ distinct zeros. On the other hand, by condition (1) of the theorem in Section 1, g has at most n distinct zeros, and therefore $n = 2$. A repetition of the argument above, only with $n = 2$ this time, shows that for every $t \in \Sigma \sim D$ there exists a $g_t \in G \sim \{0\}$ such that $Z(g_t) = \{t, t^+\}$, and therefore G is 1-Chebyshev but not Chebyshev. At this point we have all but proved the theorem: With the exception of the folkloric fact that in case X is finite the metric projection of any G is lower semi-continuous and therefore, by Michael's selection theorem, has a unique continuous selection (if and) only if G is Chebyshev, all that is missing are the examples. We shall use part (ii) of the following proposition in the construction of one of the examples. The proposition, however, is of independent interest.

PROPOSITION. (i) (A. L. Garkavi [15, final remark]). *If for some integer $n \geqslant 2$, $C(X)$ contains an n-dimensional almost Chebyshev subspace, then X has at most $2^{\aleph_0}$ isolated points.*

(ii) *If for some integer $n \geqslant 2$, $C(X)$ contains an n-dimensional vector subspace whose metric projection has a unique continuous selection, then X has at most $\aleph_0$ isolated points.*

Proof. (i) Let X_0 denote the set of isolated points of X and suppose that card $X_0 > 2^{\aleph_0}$. Let g_1 and g_2 be any two linearly independent functions in $C(X)$. g_1 is constant on some subset X_1 of X_0 of cardinality $> 2^{\aleph_0}$:

If not, $\operatorname{card} g_1^{-1}[\{r\}] \leqslant 2^{\aleph_0}$ for every $r \in \mathbb{R}$, and therefore $\operatorname{card} X_0 \leqslant \operatorname{card} \bigcup_{r \in \mathbb{R}} g_1^{-1}[\{r\}] = 2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0}$, a contradiction. A repetition of this argument with X_1 in the place of X_0 and g_2 in the place of g_1 shows that g_2 is constant on some subset X_2 of X_1 of cardinality $> 2^{\aleph_0}$. Thus, some non-trivial linear combination of g_1 and g_2 is zero on X_2, and therefore cannot be contained in any n-dimensional almost Chebyshev subspace of $C(X)$.

(ii) Since the unit sphere S^1 has no isolated points, by the embedding theorem, we are left with the case that X is homeomorphic to a subspace of some interval with split points $T(\emptyset, D^+)$, $D^+ \subset [0, 1)$, and we assume that X has been identified with this subspace. As for any compact totally ordered topological space (see the Appendix), the topology of X is the order topology, and therefore a point x in X which is neither the first nor the last point of X is an isolated point of X iff it has a predecessor (in X!) x_- and a successor x_+. A moment's reflection shows that for any such isolated point x of X,

$$\{t \in [0, 1]: \pi x_- + \tfrac{1}{2}(\pi x - \pi x_-) < t < \pi x + \tfrac{1}{2}(\pi x_+ - \pi x)\},$$

where π is the canonical projection of $T(\emptyset, D^+)$ onto the unit interval $[0, 1]$, is a non-degenerate open interval contained in $(0, 1)$, and that for any two such isolated points of X these intervals are disjoint. Thus, the set of isolated points of X is countable.

We now provide the examples required to complete the proof of the theorem. The Chebyshev examples in cases (i), (ii), and (iii) of the theorem are well known and shall not be discussed here. If for some integer $n \geqslant 2$, G is an n-dimensional Chebyshev subspace of $C(X)$, if x is a non-isolated point of X, and if h is a non-negative continuous function on X with a single zero at x, then $h \cdot G = \{h \cdot g: g \in G\}$ is an n-dimensional vector subspace of $C(X)$ which satisfies conditions (1)–(3) of the theorem in Section 1 (for (3) observe Haar's theorem and the trivial fact that the Haar condition that any non-zero function in G have at most $n-1$ distinct zeros is equivalent to the condition that for any n distinct points $x_1, ..., x_n$ of X and any n signs $\sigma_1, ..., \sigma_n$ in $\{-1, 1\}$, there exists a function g in G such that $g(x_i) = \sigma_i$ for all i), so that the metric projection of $h \cdot G$ has a unique continuous selection. By Rubinstein's theorem, $h \cdot G$ is not $(n-1)$-Chebyshev. This takes care of the non-Chebyshev examples in cases (ii) and (iii) of the theorem, and now all we need is an example in case (iv). Accordingly, suppose X is a closed subspace of some interval with split points $T(\emptyset, D^+)$, $D^+ \subset [0, 1)$, and that $\Sigma = \{t \in D^+: \text{both } t \text{ and } t^+ \text{ are in } X\}$ is uncountable. By part (ii) of the proposition, certainly $\Sigma_0 = \{t \in D^+: \text{both } t \text{ and } t^+ \text{ are isolated points of } X\}$ is countable. As we have seen in the second part of the proof of the theorem, there exists a strictly increasing function φ: $[0, 1] \to \mathbb{R}$ which is left-continuous at all points of $(0, 1]$ and right-continuous precisely at the points of $[0, 1) \sim \Sigma_0$, and there exists a unique non-decreasing and continuous function $\tilde{\varphi}$ on $T(\emptyset, D^+)$ such that $\tilde{\varphi} \mid [0, 1] = \varphi$. It is all but obvious that the sets of constancy of $\tilde{\varphi} \mid X$ which are not singletons are precisely the two-point sets $\{t, t^+\}$ for $t \in \Sigma \sim \Sigma_0$. Thus, the multiples of $\tilde{\varphi} \mid X$ together with the constant functions on X form a 2-dimensional vector subspace of $C(X)$ which satisfies conditions (1)–(3) of the theorem in Section 1. This does it.

Remarks. 1. Simple examples show that the bounds for the number of isolated points of X in parts (i) and (ii) of the proposition are both sharp.

2. It would be nice if one could prove part (ii) of the proposition without recourse to the embedding theorem.

3. Our proof of the theorem does not make use of Mairhuber's theorem; it in fact contains a proof of this theorem which compares favourably with all others known.

4. Very little seems to be known about examples in cases (ii) and (iii) of the theorem which are k-Chebyshev but not $(k-1)$-Chebyshev for some $1 \leqslant k \leqslant n-1$.

5. Before concluding this section, we feel obliged to say a word or two about why, contrary to what we related in the Introduction, our proof of the theorem does depend on Li's paper [18]. The only result of Li's paper we use in our proof is his theorem on determinants. All we actually need, however, is the special case of this theorem that G satisfies conditions (1)–(3) of the theorem in Section 1. And precisely this special case of Li's theorem we had proved well before learning of Li's paper.

APPENDIX: On Intervals with Split Points

Given a subset D^- of the interval $(0, 1]$ and a subset D^+ of the interval $[0, 1)$, we consider the disjoint (!) union

$$T(D^-, D^+) = \{t^- : t \in D^-\} \cup \{t : t \in [0, 1]\} \cup \{t^+ : t \in D^+\}.$$

It is obvious that the definition

> for every $t \in D^-$, t^- is the predecessor of t, and for every $t \in D^+$, t^+ is the successor of t,

extends the canonical order of the unit interval $[0, 1]$ to a total order for $T(D^-, D^+)$, and we provide $T(D^-, D^+)$ with the topology induced by this order, i.e., the smallest topology for $T(D^-, D^+)$ which contains all the sets $\{y \in T(D^-, D^+): y < x\}$ and $\{y \in T(D^-, D^+): x < y\}$ for some $x \in T(D^-, D^+)$. This topology is obviously Hausdorff and, using the fact that every subset of $T(D^-, D^+)$ has a supremum, it is a mildly intricate exercise to prove that it is also compact. An *interval with split points* is one of the sets $T(D^-, D^+)$ together with the order and the topology just described. We note that $T(\varnothing, \varnothing)$ is the unit interval $[0, 1]$ with its usual order and topology.

For $D^- \subset (0, 1]$ and $D^+ \subset [0, 1)$, $K(D^-, D^+)$ is the set of all non-decreasing real-valued functions on the unit interval $[0, 1]$ which are left-continuous at all points of $(0, 1] \sim D^-$ and right-continuous at all points of $[0, 1) \sim D^+$, and $\tau(D^-, D^+)$ is the smallest topology for the unit interval $[0, 1]$ which renders the functions in $K(D^-, D^+)$ continuous. It is obvious that $K(D^-, D^+)$ is a uniformly closed lattice cone (lattice cone = convex cone which is closed under the lattice operations) of non-decreasing real-valued functions on $[0, 1]$ which contains the cone $K(\varnothing, \varnothing)$ of all continuous non-decreasing real-valued functions on $[0, 1]$; the last fact implies trivially that the topology $\tau(D^-, D^+)$ contains the usual topology $\tau(\varnothing, \varnothing)$ of $[0, 1]$.

Conversely, let K be any uniformly closed lattice cone of non-decreasing real-valued functions on $[0, 1]$ which contains the cone $K(\varnothing, \varnothing)$, and let τ be the smallest topology for $[0, 1]$ which renders the functions in K continuous. J. Blatter [2, 2.7 Theorem], using the Characterization Theorem of J. Blatter and G. L. Seever [8], shows that K is the set of all non-decreasing real-valued functions on $[0, 1]$ which are τ-continuous, and this result, modulo some fiddling around with idempotent functions in K, yields immediately that $K = K(D^-, D^+)$, where D^- is the set of all points in $(0, 1]$ at which some function in K is not left-continuous and D^+ is the set of all points in $[0, 1)$ at which some function in K is not right-continuous.

A real-valued function on the unit interval $[0, 1]$ is called *regulated* if it has finite left-sided limits at all points of $(0, 1]$ and finite right-sided limits at all points of $[0, 1)$. Such functions are obviously bounded. For $D^- \subset (0, 1]$ and $D^+ \subset [0, 1)$, $A(D^-, D^+)$ is the set of all regulated real-valued functions on the unit interval $[0, 1]$ which are left-continuous at all points of $(0, 1] \sim D^-$ and right-continuous at all points of $[0, 1) \sim D^+$. It is obvious that $A(D^-, D^+)$ is a uniformly closed algebra of regulated real-valued functions on $[0, 1]$ which contains the algebra $A(\varnothing, \varnothing)$ of all continuous real-valued functions on $[0, 1]$. It is also obvious that $A(D^-, D^+)$ contains (the vector lattice of all real-valued functions of bounded variation on $[0, 1]$ which are left-continuous at all points of $(0, 1] \sim D^-$ and right-continuous at all points of $[0, 1) \sim D^+$) $K(D^-, D^+) - K(D^-, D^+)$. It is not at all obvious, however, that $A(D^-, D^+) = \mathrm{cl}(K(D^-, D^+) - K(D^-, D^+))$ (cl = uniform closure). To see this, let $f \in A(D^-, D^+)$ and let

$\varepsilon > 0$. Since f is regulated, for every $t \in (0, 1]$ there exists $a(t) \in (0, t)$ such that $|f(r) - f(s)| \leqslant \varepsilon$ for all $r, s \in [a(t), t)$, and for every $t \in [0, 1)$ there exists $b(t) \in (t, 1)$ such that $|f(r) - f(s)| \leqslant \varepsilon$ for all $r, s \in (t, b(t)]$. By compactness, there exists a finite number $l \geqslant 1$ of points $t_1, ..., t_l$ in $(0, 1)$ such that

$$[0, 1] = [0, b(0)) \cup \bigcup_{j=1}^{l} (a(t_j), b(t_j)) \cup (a(1), 1].$$

Set $t_0 = 0$ and $t_{l+1} = 1$, call the distinct points among $t_0, ..., t_{l+1}$, $a(t_1), ..., a(t_{l+1})$, and $b(t_0), ..., b(t_l)$ in increasing order $c_0, ..., c_{k+1}$, and define a (linear spline) function g: $[0, 1] \to \mathbb{R}$ by stipulating that g coincide with f at the points $c_0, ..., c_{k+1}$ and $\frac{1}{2}(c_0 + c_1), ..., \frac{1}{2}(c_k + c_{k+1})$, and that for $i = 1, ..., k+1$, g be

linear and continuous on $[c_{i-1}, \frac{1}{2}(c_{i-1} + c_i)]$ if $c_{i-1} \notin D^+$,
constant on $(c_{i-1}, \frac{1}{2}(c_{i-1} + c_i)]$ if $c_{i-1} \in D^+$,
constant on $[\frac{1}{2}(c_{i-1} + c_i), c_i)$ if $c_i \in D^-$, and
linear and continuous on $[\frac{1}{2}(c_{i-1} + c_i), c_i]$ if $c_i \notin D^-$.

It is easily seen that g is a linear combination of the functions

$$\varphi_i(t) = t^i, \qquad 0 \leqslant t \leqslant 1,\ i = 0, 1,$$

$$\psi_{i,1}(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant \frac{1}{2}(c_{i-1} + c_i), \\ t - \frac{1}{2}(c_{i-1} + c_i) & \text{if } \frac{1}{2}(c_{i-1} + c_i) < t \leqslant 1, \end{cases} \quad i = 1, ..., k+1,$$

$$\psi_{i,2}(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant c_i, \\ t - c_i & \text{if } c_i < t \leqslant 1, \end{cases} \quad i = 1, ..., k,$$

$$\psi_{i,2}^-(t) = \begin{cases} 0 & \text{if } 0 \leqslant t < c_i, \\ 1 & \text{if } c_i \leqslant t \leqslant 1, \end{cases} \quad i \in \{1, ..., k+1\} \text{ such that } c_i \in D^-,$$

and

$$\psi_{i,2}^+(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant c_i, \\ 1 & \text{if } c_i < t \leqslant 1, \end{cases} \quad i \in \{0, ..., k\} \text{ such that } c_i \in D^+.$$

Since all these functions are in $K(D^-, D^+)$, g is in $K(D^-, D^+) - K(D^-, D^+)$, and therefore we'll be through if we can show that $\|f - g\| \leqslant \varepsilon$: Let $i \in \{0, ..., k+1\}$. Clearly, at least one of

$$c_i \in [t_j, b(t_j)) \qquad \text{for some} \quad j \in \{0, ..., l\}$$

and

$$c_i \in (a(t_j), t_j] \qquad \text{for some} \quad j \in \{1, ..., l+1\}$$

occurs. Suppose the latter. In this case $i \neq 0$ and $a(t_j) \leqslant c_{i-1} < c_i \leqslant t_j$. By the construction of $a(t_j)$,

$$|f(r) - f(s)| \leqslant \varepsilon \qquad \text{for all} \quad r, s \in [c_{i-1}, c_i).$$

And by the construction of g, if t is one of c_{i-1}, $\frac{1}{2}(c_{i-1} + c_i)$, and c_i, then $g(t) = f(t)$; if $t \in (c_{i-1}, \frac{1}{2}(c_{i-1} + c_i))$, then

$$\begin{aligned} c_{i-1} \in D^+ \Rightarrow g(t) = g(\tfrac{1}{2}(c_{i-1} + c_i)) \Rightarrow |f(t) - g(t)| \\ = |f(t) - g(\tfrac{1}{2}(c_{i-1} + c_i))| \\ = |f(t) - f(\tfrac{1}{2}(c_{i-1} + c_i))| \leqslant \varepsilon, \end{aligned}$$

and

$$c_{i-1} \notin D^+ \Rightarrow g(t) = \alpha g(c_{i-1}) + (1-\alpha)\, g(\tfrac{1}{2}(c_{i-1} + c_i))$$

for some $\alpha \in (0, 1) \Rightarrow |f(t) - g(t)| \leqslant \alpha\, |f(t) - g(c_{i-1})| + (1-\alpha)\, |f(t) - g(\frac{1}{2}(c_{i-1} + c_i))| = \alpha\, |f(t) - f(c_{i-1})| + (1-\alpha)\, |f(t) - f(\frac{1}{2}(c_{i-1} + c_i))| \leqslant \alpha\varepsilon + (1-\alpha)\varepsilon = \varepsilon$; and if $t \in (\frac{1}{2}(c_{i-1} + c_i), c_i)$, then

$$\begin{aligned} c_i \in D^- \Rightarrow g(t) = g(\tfrac{1}{2}(c_{i-1} + c_i)) \Rightarrow |f(t) - g(t)| \\ = |f(t) - g(\tfrac{1}{2}(c_{i-1} + c_i))| \\ = |f(t) - f(\tfrac{1}{2}(c_{i-1} + c_i))| \leqslant \varepsilon, \end{aligned}$$

and

$$c_i \notin D^- \Rightarrow g(t) = \alpha g(\tfrac{1}{2}(c_{i-1} + c_i)) + (1-\alpha)\, g(c_i)$$

for some $\alpha \in (0, 1) \Rightarrow |f(t) - g(t)| \leqslant \alpha\, |f(t) - g(\frac{1}{2}(c_{i-1} + c_i))| + (1-\alpha)\, |f(t) - g(c_i)| = \alpha\, |f(t) - f(\frac{1}{2}(c_{i-1} + c_i))| + (1-\alpha)\, |f(t) - f(c_i)| \leqslant$ (observe that $c_i \notin D^-$ and therefore $|f(r) - f(s)| \leqslant \varepsilon$ for all $r, s \in [c_{i-1}, c_i]$) $\leqslant \alpha\varepsilon + (1-\alpha)\varepsilon = \varepsilon$.

Thus, $|f(t) - g(t)| \leqslant \varepsilon$ for all $t \in [c_{i-1}, c_i]$ if $c_i \in (a(t_j), t_j]$ for some $j \in \{1, ..., l+1\}$. By symmetry, $|f(t) - g(t)| \leqslant \varepsilon$ for all $t \in [c_i, c_{i+1}]$ if $c_i \in [t_j, b(t_j))$ for some $j \in \{0, ..., l\}$. We're through.

Now let A be any uniformly closed algebra of regulated functions on $[0, 1]$ which contains the algebra $A(\varnothing, \varnothing)$ and which is generated by its cone of non-decreasing functions, i.e., $A = \mathrm{cl}(K - K)$ with $K = \{f \in A : f \text{ non-decreasing}\}$. We have seen already that $K = K(D^-, D^+)$, where D^- is the set of all points in $(0, 1]$ at which some function in K is not left-continuous and D^+ is the set of all points in $[0, 1)$ at which some function in K is not right-continuous. And we have also seen already that $\mathrm{cl}(K(D^-, D^+) - K(D^-, D^+)) = A(D^-, D^+)$. These two facts imply first that D^- also is the set of all points in $(0, 1]$ at which some function in A is not left-continuous and D^+ the set of all points in $[0, 1)$ at which some function in A is not right-continuous, and then that $A = \mathrm{cl}(K - K) = \mathrm{cl}(K(D^-, D^+) - K(D^-, D^+)) = A(D^-, D^+)$.

With the aid of the above characterizations of the cones $K(D^-, D^+)$ and the algebras $A(D^-, D^+)$ one deduces from the results of J. Blatter [2, 3] and J. Blatter and G. L. Seever [8] the two interpretations of intervals with split points alluded to in the Introduction.

A *totally ordered topological space* is a set X provided with a total order and a topology which contains the topology induced by that order. An *order compactification* of a totally ordered topological space X is a pair $(Y, \varkappa)$ consisting of a compact totally ordered space Y and a mapping $\varkappa: X \to Y$ such that

$\varkappa$ is a topological embedding,

$\varkappa$ is an order embedding, i.e., $\varkappa(x) \leqslant \varkappa(y)$ iff $x \leqslant y$, and

$\varkappa[X]$ is dense in Y.

Two order compactifications $(Y_1, \varkappa_1)$ and $(Y_2, \varkappa_2)$ of a totally ordered topological space X are *equivalent* if there exists a mapping $\varphi: Y_1 \to Y_2$ such that

φ is a topological isomorphism,

φ is an order isomorphism, and

$\varphi \circ \varkappa_1 = \varkappa_2$.

THEOREM. (i) *The totally ordered topological space* $[0, 1]$ *with the usual order and some topology* τ *which contains the usual topology has an order compactification iff* τ *is one of the topologies* $\tau(D^-, D^+)$;

(ii) *If* $D^- \subset (0, 1]$ *and* $D^+ \subset [0, 1)$, *then* $[0, 1]$ *with the usual order and the topology* $\tau(D^-, D^+)$ *has, modulo equivalence, a unique order compactification, namely the interval with split points* $T(D^-, D^+)$ *together with the inclusion mapping, and moreover* $\{f \mid [0, 1]: f \in C(T(D^-, D^+))$ *non-decreasing*$\} = K(D^-, D^+)$.

Let A be a commutative real Banach algebra which has an identity and which satisfies the Arens conditions

$$1 + f^2 \text{ is invertible and } \|f^2\| = \|f\|^2 \text{ for all } f \in A.$$

The *Gelfand space of* A is the set Γ_A of all non-zero multiplicative linear functionals on A topologized as a subspace of the product space $\mathbb{R}^A$. Γ_A is a non-empty compact Hausdorff topological space. For $f \in A$, the *Gelfand transform of f* is the function $\hat{f}\colon \Gamma_A \to \mathbb{R}$ defined by

$$\hat{f}(\gamma) = \gamma(f), \qquad \gamma \in \Gamma_A.$$

The *Gelfand transform of A*, $f \mapsto \hat{f}$, is a multiplicative linear isometry of A onto $C(\Gamma_A)$.

THEOREM. *If $D^- \subset (0, 1]$ and $D^+ \subset [0, 1)$ and if for each $x \in T(D^-, D^+)$ the functional $\delta_x\colon A(D^-, D^+) \to \mathbb{R}$ is defined by*

$$\delta_x(f) = \begin{cases} f(t-) & \text{if } x = t^- \text{ for some } t \in D^-, \\ f(t) & \text{if } x = t \in [0, 1], \\ f(t+) & \text{if } x = t^+ \text{ for some } t \in D^+, \end{cases} \qquad f \in A(D^-, D^+),$$

then the mapping $x \mapsto \delta_x$ is a homeomorphism of the interval with split points $T(D^-, D^+)$ onto the Gelfand space $\Gamma_{A(D^-, D^+)}$ of the algebra $A(D^-, D^+)$, and therefore the extension mapping $f \mapsto \hat{f} \circ \delta$ is a multiplicative linear isometry of $A(D^-, D^+)$ onto $C(T(D^-, D^+))$.

Remarks. 1. There do exist topologies for $[0, 1]$ between $\tau(\emptyset, \emptyset)$ and $\tau((0, 1], [0, 1))$ other than the topologies $\tau(D^-, D^+)$, and there do exist uniformly closed algebras of bounded functions between $A(\emptyset, \emptyset)$ and $A((0, 1], [0, 1))$ other than the algebras $A(D^-, D^+)$.

2. In the special case that $D^- = (0, 1]$ and $D^+ = [0, 1)$, the last theorem was first proved by S. Berberian [1]; see, however, the discussion in J. Blatter [3].

Note added in proof. 1. The referee points out that W. Li ("Various continuities of metric projections in $C_0(T, X)$," *J. Approx. Theory* **57** (1989), 150–168) also discovered a link between the uniqueness of continuous selections and the almost Chebyshev property and, in particular, proved the proposition in Section 1 in the case that X is metrizable.

2. The referee also points out that the theorem in Section 1 remains true if $C(X)$ is replaced by $C_0(X)$, X a locally compact Hausdorff topological space, and that it would be interesting to know if the theorem in Section 2 can be extended in the same way; that both Haar's theorem and Mairhuber's theorem can be so extended is due to, respectively, R. R. Phellps ("Uniqueness of Hahn–Banach extensions and unique best approximations," *Trans. Amer. Math. Soc.* **95** (1960), 238–255) and J. A. Lutts ("Topological spaces which admit unisolvent systems," *Trans. Amer. Math. Soc.* **111** (1964), 440–448).

REFERENCES

1. S. BERBERIAN, The character space of the algebra of regulated functions, *Pacific J. Math.* **74** (1978), 15–36 (MR 81 b: 46067).
2. J. BLATTER, Order compactifications of totally ordered topological spaces, *J. Approx. Theory* **13** (1975), 56–65 (MR 50 #11153).

3. J. BLATTER, The Gelfand space of the Banach algebra of Riemann integrable functions, *in* "Approximation Theory III (Proceedings, Conference at the University of Texas, Austin, TX, 1980)," pp. 229–232, Academic Press, New York, 1980 (MR 82 c: 46069).
4. J. BLATTER, Intervalos com pontos "splitados," *in* "Atas, 22° Sem. Bras. Análise, Univ. Fed. Rio de Janeiro, 1985," pp. 525–531.
5. J. BLATTER, Unique continuous selections for metric projections in $C(X)$, *Abstracts Amer. Math. Soc.* **9** (1988), 144–145.
6. J. BLATTER AND L. SCHUMAKER, The set of continuous selections of a metric projection in $C(X)$, *J. Approx. Theory* **36** (1982), 141–155 (MR 83 k: 41021).
7. J. BLATTER AND L. SCHUMAKER, Continuous selections and maximal alternators for spline approximation, *J. Approx. Theory* **38** (1983), 71–80 (MR 84 j: 41013).
8. J. BLATTER AND G. L. SEEVER, Interposition and lattice cones of functions, *Trans. Amer. Math. Soc.* **222** (1976), 65–96 (MR 55 #11013).
9. A. L. BROWN, On continuous selections for metric projections in spaces of continuous functions, *J. Funct. Anal.* **8** (1971), 431–449 (MR 45 #5725).
10. A. L. BROWN, An extension to Mairhuber's theorem. On metric projections and discontinuity of multivariate best uniform approximation, *J. Approx. Theory* **36** (1982), 156–172 (MR 83 k: 41022).
11. P. C. CURTIS, JR., n-Parameter families and best approximation, *Pacific J. Math.* **9** (1959), 1013–1027 (MR 21 #7385).
12. F. DEUTSCH AND G. NÜRNBERGER, Weakly interpolating subspaces, *Numer. Funct. Anal. Optim.* **5** (1982/1983), 267–288 (MR 84 h: 41045).
13. A. L. GARKAVI, On Čebyšev and almost Čebyšev subspaces, *Dokl. Akad. Nauk SSSR* **149** (1963), 1250–1252 (= *Soviet Math. Dokl.* **4** (1963), 532–534) (MR 26 #6737).
14. A. L. GARKAVI, On Čebyšev and almost Čebyšev subspaces, *Izv. Akad. Nauk SSSR Ser. Mat.* **28** (1964), 799–818 (= *Amer. Math. Soc. Transl.* **96** (1970), 153–175) (MR 29 #2635).
15. A. L. GARKAVI, Almost Čebyšev systems of continuous functions, *Izv. Vyssh. Uchebn. Zaved. Mat.* **45** (1965), 36–44 (= *Amer. Math. Soc. Transl.* **96** (1970), 177–187) (MR 32 #1550).
16. A. HAAR, Die Minkowskische Geometrie und die Annäherung an stetige Funktionen, *Math. Ann.* **78** (1918), 294–311.
17. A. J. LAZAR, D. E. WULBERT, AND P. D. MORRIS, Continuous selections for metric projections, *J. Funct. Anal.* **3** (1969), 193–216 (MR 39 #3288).
18. W. LI, Continuous selections of metric projections and regular weakly interpolating subspaces, preprint, 1987.
19. J. C. MAIRHUBER, On Haar's theorem concerning Chebychev approximation problems having unique solutions, *Proc. Amer. Math. Soc.* **7** (1956), 609–615 (MR 18-125).
20. G. NÜRNBERGER AND M. SOMMER, Continuous selections in Chebyshev approximation, *in* "Parametric Optimization and Approximation (Proceedings, Conference, Oberwolfach, 1983)," pp. 248–263, Birkhäuser, Basel, 1985.
21. G. S. RUBINSTEIN, On a method of investigation of convex sets, *Dokl. Akad. Nauk SSSR* **102** (1955), 451–454 (MR 17-185).
22. I. J. SCHOENBERG AND C. T. YANG, On the unicity of problems of best approximation, *Ann. Mat. Pura Appl.* **54** (1961), 1–12 (MR 25 #5324).
23. K. SIEKLUCKI, Topological properties of sets admitting the Tschebycheff systems, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astron. Phys.* **6** (1958), 603–606 (MR 20 #6625).